J Stat Phys (2010) 139: 1–26
DOI 10.1007/s10955-010-9941-8

On the Number of Entangled Clusters

Mahshid Atapour · Neal Madras

Received: 28 July 2009 / Accepted: 19 February 2010 / Published online: 5 March 2010

Abstract We prove that the number of entangled clusters with N edges in the simple cubic lattice grows exponentially in N. This answers an open question posed by Grimmett and Holroyd (Proc. Lond. Math. Soc. 81:485–512, 2000). Our result has immediate implications for entanglement percolation: we obtain an improved rigorous lower bound on the critical probability, and we prove that the radius of the entangled component of the origin has exponentially decaying tail when p is small.

Keywords Entanglement percolation · Entangled cluster · Lattice animal · Polymer · Linking · Critical probability

1 Introduction

Topological entanglements play a subtle but important role in the statistical mechanics of macromolecules. Experiments show that the synthesis of ring polymers in solution can result in interlocked sets of rings, called catenanes (e.g. [30]). Entanglements influence the elasticity properties of polymer networks [6, 26]. Theorists have proposed the existence of large collections of mutually interlinked polymer rings, called Olympic ring networks, which could have unusual physical properties [4, 29]. See [27] for surveys on these and other aspects of molecular topology.

Polymers in solution can form very large networks, typically either by polymerization of monomers with functionality of more than two (creating *branched* polymers) or else by cross-linking between linear polymers (e.g. [3]). The formation of an essentially infinite network (a *gel*) is often modeled on the lattice by percolation (see below). But it is also possible for large networks to form by the physical (topological) interlinking of many smaller branched polymers that contain cycles [27].

M. Atapour · N. Madras (✉)
Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, Ontario M3J 1P3, Canada
e-mail: madras@mathstat.yorku.ca

M. Atapour
e-mail: atapour@mathstat.yorku.ca

The purpose of this paper is to answer rigorously some basic questions about large entangled clusters on the simple cubic lattice, in terms of understanding both their entropy (i.e., how many are there?) and the entanglement percolation/gelation model. Everything in this paper takes place in three dimensions.

To introduce the problems mathematically, consider bond percolation in the simple cubic lattice. That is, for a given probability p, each edge of the lattice is independently declared to be either "open" (with probability p) or "closed" (with probability $1-p$). It is well known (see [9] for a comprehensive treatment of percolation) that the lattice has a critical probability p_c, strictly between 0 and 1, such that (with probability 1) the subgraph of open edges has a unique infinite connected component if $p > p_c$ and has no infinite connected component if $p < p_c$. But when p is just a bit less than p_c, is it possible that infinitely many finite components link up with one another to form an "infinite entangled graph"? Intuitively, a subgraph of the simple cubic lattice is entangled if it is impossible to deform space continuously so that part (but not all) of the subgraph lies inside a ball (and none of the deformed subgraph touches the surface of the ball). Numerical investigations by [17] indicated that there is an "entanglement threshold" p_e which is less than p_c by about 1.8×10^{-7}. However, Kantor and Hassold only looked for a special subclass of entangled graphs, namely those that had two components that touched opposite faces of a large finite box. Hence, their estimate may be best viewed as a lower bound for $p_c - p_e$.

The paper is organized as follows. Section 1.1 describes previous mathematical progress on this topic, including a brief discussion of the definition of p_e. It also presents our results, including our main theorem, which says that the number (up to translation) of clusters with N edges grows exponentially in N. Section 1.2 presents the intuition behind our main theorem. Section 1.3 describes some related models with different behaviours from ours. Section 2 establishes the terminology and notation that we use throughout the paper. Section 3 introduces the concept of *block-cluster*. Whereas a cluster can be viewed as a collection of edges and vertices, a block-cluster is a collection of edges, vertices, and "boxes" in $\mathbf{R}^3$. Section 4 defines a transformation on clusters that puts boxes around the cycles of the clusters. We shall be using the intersection of a (connected) component with the box of another component as a surrogate for entanglement of components. Section 5 presents our main technical result: we show that it is possible to recursively add a controlled number of edges to an entangled cluster so as to join up all the connected components. Section 6 contains the proofs of the theorems stated in Sect. 1.1. Section 7 presents our conclusions and a discussion of some consequences of our results.

1.1 Previous Work and New Results

The formal mathematical framework of infinite entanglement graphs has been addressed by Grimmett and Holroyd [10]. They show that the definition of (infinite) entangled graph is not unique, but that there are two "extremal" definitions, roughly corresponding to "free" and "wired" boundary conditions. (The simulations of [17] used an intermediate definition: for configurations in a large finite box, the top and bottom of the box were "wired" but the sides were "free".) The two extremal definitions give rise to two (possibly equal) critical probabilities for entanglement, which we call p_e^0 and p_e^1 (for free and wired boundary conditions, respectively). The precise definitions are given in Sects. 2.2 and 2.3. From first principles it follows that $p_e^1 \le p_e^0 \le p_c$. The strict inequality $p_e^0 < p_c$ predicted by Kantor and Hassold [17] was proven rigorously by the work of [2, 15]. For ordinary percolation we know that p_c is strictly between 0 and 1; Holroyd [14] proved the analogue for every possible definition of p_e, by giving the explicit bound $p_e^1 \geq 1/15616$. Thus we know

$$0 < p_e^1 \leq p_e^0 < p_c < 1. \tag{1}$$

Häggström [12] proved that the infinite entangled component is unique above the critical probability (for any definition of entanglement). It is not known yet whether p_e^1 equals p_e^0.

Next, consider the set $\mathcal{F}$ of finite entangled subgraphs of the simple cubic lattice (the precise definition appears in Sect. 2.2). For $N \geq 1$, let e_N be the number of subgraphs in $\mathcal{F}$ that have exactly N edges and contain the origin. Grimmett and Holroyd [10] proved that $e_N \leq \exp(bN + \frac{3}{8}N \log N)$ for some constant b, but could not determine the leading order of the true growth rate of e_N. The analogues of finite entangled subgraphs in ordinary percolation are finite connected subgraphs, also called "lattice animals". Let a_N be the number of connected subgraphs of the simple cubic lattice that have exactly N edges and contain the origin. It is known that the number of lattice animals grows exponentially, in the sense that

$$\lambda := \lim_{N\to\infty} a_N^{1/N} \tag{2}$$

exists and is finite. The proof of this has two parts. First, the existence of the limit follows from a concatenation argument [19, 20]. Secondly, proof of finiteness requires an exponential upper bound—for approaches to obtaining this bound, see [19] (described in general dimensions in Lemma 6.1 of [16]) and Lemma 1 of [18] (described in Sect. 4.2 of [9]). The main result of this paper is the following analogous assertion for finite entangled graphs.

Theorem 1 *For every $N \geq 1$,*

$$e_N \leq 4^N a_{2N}. \tag{3}$$

Furthermore,

$$\lambda_e := \lim_{N\to\infty} e_N^{1/N} \quad \textit{exists and satisfies} \quad \lambda < \lambda_e \leq 4\lambda^2, \tag{4}$$

where λ is defined in (2).

To provide some insight into the subtlety of this result, Sect. 1.3 discusses a class of finite subgraphs that seems to be similar to entangled graphs but grows faster than exponentially.

The bound of Theorem 1 translates fairly directly into the bound $p_e^0 \geq 1/(4\lambda^2)$. Some additional theory is required to obtain the same bound on p_e^1.

Theorem 2

$$p_e^0 \geq p_e^1 \geq \frac{1}{\lambda_e} \geq \frac{1}{4\lambda^2}.$$

We remark that the proof of strict positivity of p_e^1 in [14] works with the 'duality' relationship of plaquette percolation, by showing that large dual surfaces must enclose the origin when the density of plaquettes is close to 1. In contrast, our proof works with the entangled clusters directly.

Grimmett and Holroyd [10] also considered the maximal open entangled graph containing the origin, which we shall call $C(\mathcal{E}_0)$ (see Sect. 2 for precise definitions). In particular, they proved that when p is sufficiently small, the probability measure P_p of bond percolation satisfies

$$P_p\{\mathrm{diam}(C(\mathcal{E}_0)) > r\} \leq \exp(-r/h(r))$$

where $\mathrm{diam}(C(\mathcal{E}_0))$ is the diameter of $C(\mathcal{E}_0)$ and h is a function that grows proportionally to some number of iterates of logarithm. Theorem 1 allows us to improve this to exponential decay for small p.

Theorem 3 *Fix* $p < 1/\lambda_e$. *Then* $P_p\{\mathrm{diam}(C(\mathcal{E}_0)) > r\}$ *decays exponentially in* r. *More precisely,*

$$\limsup_{N\to\infty} P_p\{C(\mathcal{E}_0) \textit{ has at least } N \textit{ edges}\}^{1/N} \le p\lambda_e. \tag{5}$$

We observe that Theorem 3 also holds if we replace $\mathcal{E}_0$ by $\mathcal{E}_1$ (because below $1/\lambda_e$, the components $C(\mathcal{E}_0)$ and $C(\mathcal{E}_1)$ are both in $\mathcal{F}$, hence $C(\mathcal{E}_0) = C(\mathcal{E}_1)$). We also note that Holroyd [15] proved that if $p_e^1 \ne p_e^0$, then the left hand side of (5) equals 1 whenever p is between p_e^1 and p_e^0.

Finally we mention the following result which improves the upper bound of Theorem 1 for a special class of entangled graphs. Motivated by work in the chemical physics literature (e.g. [4]), we define an *olympic ring network* to be an entangled graph in which every connected component is a *self-avoiding polygon* (i.e., a simple closed curve in the lattice). An example is shown in Fig. 1 below.

Theorem 4 *For every* $N \ge 1$, *let* r_N *be the number of olympic ring networks containing the origin and having exactly* N *edges. Then*

$$\lim_{N\to\infty} r_N^{1/N} \le \lambda^2, \tag{6}$$

where λ *is defined in* (2).

For a different model of random olympic ring networks, see [5].

What do the bounds of Theorems 1, 2 and 4 imply numerically? Gaunt and Ruskin [7] used exact enumeration to obtain the estimate $\lambda \approx 10.62$, which suggests the bounds $\lim_{N\to\infty} e_N^{1/N} \le 451.14$, $p_e^1 \ge 0.002216$, and $\lim_{N\to\infty} r_N^{1/N} \le 112.79$. We have no reason to think that these bounds are particularly good (note that p_c is close to 0.249, according to [1]). From the rigorous point of view, we can prove that $\lambda \le 5^5/4^4 \approx 12.21$ (see Proposition 20), which leads to the rigorous bounds

$$\lim_{N\to\infty} e_N^{1/N} \le \frac{5^{10}}{4^7} \approx 596.05 \tag{7}$$

$$p_e^0 \ge p_e^1 \ge \frac{4^7}{5^{10}} > \frac{1}{597} \approx 0.00168 \tag{8}$$

$$\lim_{N\to\infty} r_N^{1/N} \le \frac{5^{10}}{4^8} \approx 149.01 \tag{9}$$

The bound (8) improves on the only previous rigorous bound, $p_e^1 \ge 1/15616$ [14], by a factor of about 26.

1.2 Intuition Behind Theorem 1

We now give a heuristic explanation of why the bound (3) of Theorem 1 "should" be true.

Consider an entangled subgraph G of the simple cubic lattice with N edges. (Also suppose G contains the origin.) Suppose there exists a set A of k edges (not in G) such that the graph determined by $G \cup A$ is connected. Given the resulting animal $G \cup A$, as well as N and k, there are at most $\binom{N+k}{k}$ possibilities for what the original animal G could have been, and this number is less than 2^{N+k}. Suppose we can show that there is a constant t such that we can always take $k \leq tN$. Then we could deduce that

$$e_N \leq 2^{N+tN} a_{N+tN}. \tag{10}$$

Since we know that a_k grows exponentially, this would prove an exponential bound on e_N.

How can we bound k in terms of N? Let the connected components of G be $g_1, \dots, g_r$. For each $i = 1, \dots, r$, consider the convex hull of g_i (i.e., the smallest convex set in $\mathbf{R}^3$ containing the vertices of g_i). The convex hull of g_i must intersect some other component g_j (if not, then we could shrink g_i to a point inside its convex hull and then remove it through the spaces in the cubic lattice, which contradicts the assumption that G is entangled). So there is a path π_i in the simple cubic lattice inside the convex hull of g_i from a point of g_i to a point of $G \setminus g_i$. How many edges does π_i need? The diameter of the convex hull is clearly enough, which is bounded by the number of edges in g_i. But we should be able to do better than the latter bound. If g_i is a single cycle, then the diameter of g_i's convex hull is bounded by *half* the number of edges in g_i. More generally, assume we can find a cycle of g_i that is "responsible for its entanglement", in the (weak) sense that its convex hull intersects a point of $G \setminus g_i$. Under this assumption, the number of edges needed for π_i is at most half the number of edges in g_i. Thus it seems reasonable that the number of edges needed for $\pi_1, \dots, \pi_r$ in total is at most $N/2$. (Indeed, the number $N/2$ is correct: see the Remark following the proof of Proposition 14 as well as Proposition 12(a). However, there need not be a "cycle responsible for entanglement"—e.g. the large "loop" in the top sketch of Fig. 5 can be responsible for entanglement, but it is not a cycle.)

Let G_1 be the union of G and all the π_i's. Clearly G_1 has fewer components than G, but G_1 may not be connected. For example, consider a collection of small cycles $c_1, \dots, c_m$ which are linked together cyclically (i.e., c_j is linked to $c_{j\pm1}$ for each j, and c_1 is linked to c_m). If we call this configuration a "bracelet", then suppose G consists of two bracelets linked with one another as in Fig. 1. No cycle in one bracelet has a convex hull that contains any point of the other bracelet. Thus G_1 will have two components.

Although G_1 may be disconnected, G_1 must be entangled. So we repeat the above procedure on G_1 (look at the convex hulls, find paths joining components, etc.), obtaining a new entangled graph G_2. The number of edges needed in all the new paths used to make G_2 from G_1 can be bounded by the sum of diameters of all connected components of G_1. If each component of G_1 came from a bracelet-like configuration in G, then the diameter

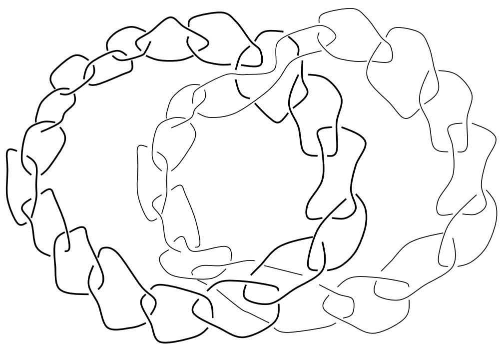


Fig. 1 Sketch of two bracelets that form a single entangled cluster

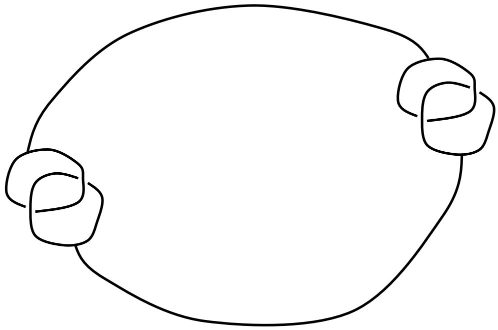


Fig. 2 A different kind of bracelet

of a component is at most one *quarter* of the number of edges of the bracelet (observe that a typical plane slicing through a bracelet intersects the bracelet in at least two distinct cycles, hence in at least four edges). Repeat this procedure to construct $G_3, G_4, \ldots$ until we obtain a graph G_j that is connected. The total number of edges in $G_i \setminus G_{i-1}$ is at most $N/2^i$. Therefore the number of edges added to G to obtain G_j is st most

$$\frac{N}{2} + \frac{N}{4} + \cdots + \frac{N}{2^i} + \cdots$$

which is N. That is, we can take $t = 1$ in (10).

This is the heuristic argument. In fact, this gives the correct final answer: Corollary 16 says that every entangled cluster with N edges is contained in an animal with at most $2N$ edges. But the argument in the preceding paragraph is too simplistic. For example, Fig. 2 shows that a bracelet's diameter could be close to half the number of edges rather than one quarter. Also, when there are many connected components, it is not clear how to characterize the levels of "bracelets". We avoid this problem by having Proposition 15 do the bookkeeping of how many edges must be added at each step. While the details of the preceding heuristic argument are not true, the overall idea is correct in spirit, and we constructed our proof with this argument as a guide. The fact that the heuristic argument produces the correct answer suggests that its hypothesized configurations are a kind of worst-case situation.

1.3 Different Behaviours in Other Models

To see why a finite exponential bound (in the number of edges) on the number of entangled clusters is not "obvious", we first consider a set of subgraphs of the cubic lattice that we shall call "caged clusters." The connected components of caged clusters exhibit a kind of geometric inseparability rather than topological inseparability. It seems reasonable to guess that if the number of entangled clusters grows exponentially (in the number of edges) then so should the number of caged clusters. However, we shall show now that the number of caged clusters grows *faster* than exponentially. In our opinion, this observation makes the exponential growth of the number of entangled clusters less "obvious."

For each integer $n \geq 1$, let $\partial B(n)$ denote the subgraph of the simple cubic lattice consisting of all edges and vertices that are contained in the boundary of the cube $[-n, n]^3$.

Consider the cluster $H := \partial B(1) \cup \partial B(n)$ (with $n > 1$). This cluster is disconnected, but if we view the two components of H as rigid subsets of $\mathbf{R}^3$ that can be translated and rotated but not stretched, shrunk, or bent (in particular, angles between edges may not change), then $\partial B(n)$ acts like a cage that keeps the small cube "inside": any continuous rigid motion of

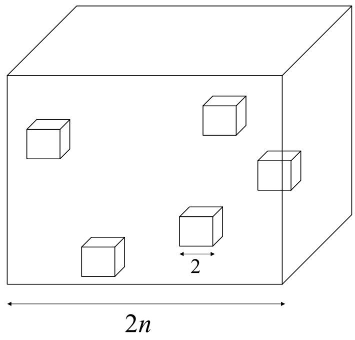


Fig. 3 Sketch of a "caged cluster." Many edges in the faces of the cubes are not shown

$\partial B(1)$ cannot leave $[-n, n]^3$ without intersecting $\partial B(n)$. In this sense, we say that H is a "caged cluster". We shall consider caged clusters consisting of a large $\partial B(n)$ enclosing many disjoint translations of $\partial B(1)$ (see Fig. 3). For large even n, partition $[-n, n]^3$ into Kn^2 columnar rectangular blocks of dimensions $4 \times 4 \times 2n$. Observe that $\partial B(n)$ has $48n^2$ edges. We can create a caged cluster with $N := 48n^2 + 48(Kn^2)$ edges by taking $\partial B(n)$ and one translate of $\partial B(1)$ in the interior of each columnar block. This shows that

$$CC_N \geq (K_2 n)^{Kn^2} = \left(K_3\sqrt{N}\right)^{K_4 N},$$

and hence that $\limsup_{N\to\infty} CC_N^{1/N} = \infty$. That is, *the number of caged clusters with N edges containing the origin grows faster than exponentially in N.*

We also mention two other interesting percolation-type models that, unlike entanglement percolation, have critical probabilities equal to 0: bootstrap percolation [28] and a tension model with random holes [25]. The analysis in [25] includes a process reminiscent of our method of joining up connected clusters that are topologically linked (as outlined in Sect. 1.2). In [25], two clusters are joined up if one cluster intersects the convex hull of the other, and the result is that these growing clusters eventually fill up all of space. This criterion for joining up is similar to our relation of one cluster "covering" the other (see the end of Sect. 4). It is also similar to the criterion used in [10] that led them to their bound $e_N \leq \exp(bN + \frac{3}{8}N\log N)$. A key difference between those arguments and ours is that we ultimately require a form of "mutual covering", either in the form of two clusters that cover each other, or in the form of a "directed cycle" of covering (e.g. cluster A covers cluster B, cluster B covers cluster C, and cluster C covers cluster A). This kind of mutual covering is not required in the model of [25], and was not used in [10]. Also, notice that in our example of caged clusters, the large cube "covers" all of the little cubes, but nothing covers the large cube. Thus it appears that mutual covering should be an essential part of the argument that λ_e is finite.

2 Definitions and Notation

2.1 Geometric and Graph-Theoretical Terminology

For a set S, $|S|$ denotes the cardinality of S. For $x \in \mathbf{R}^3$, write $x = (x_1, x_2, x_3)$.

A *box* is a subset of $\mathbf{R}^3$ of the form $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ (where $-\infty < a_i \leq b_i < +\infty$, $i = 1, 2, 3$). The *diameter* of the box $\beta = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ equals

$\sum_{i=1}^{3} |b_i - a_i|$ and is written diam(β). For bounded $S \subset \mathbf{R}^3$, let Box(S) be the smallest box that contains S.

A *lattice box* is a box $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ in $\mathbf{R}^3$ such that a_1, a_2, a_3, b_1, b_2, and b_3 are all integers, and $a_i = b_i$ for at most one i in $\{1, 2, 3\}$.

For a graph G, let $E(G)$ be the set of edges of G, and let $V(G)$ be the set of vertices of G. Two or more edges $e_1, \ldots, e_k$ are called *multiple edges* if they all have the same two endpoints.

We shall write $\mathbf{Z}_G^3$ to denote the simple cubic lattice, i.e. the infinite graph embedded in $\mathbf{R}^3$ whose vertices are the points of $\mathbf{Z^3}$ and whose edges are all unordered pairs $\{x, y\}$ of vertices that are exactly distance 1 apart. For each edge $e = \{x, y\}$ in $E(\mathbf{Z}_G^3)$, we write $\langle e \rangle$ to denote the corresponding line segment of unit length whose endpoints are x and y. The *interior points* of an edge e are the points of $\langle e \rangle$ that are not its endpoints. A *cluster* is any subgraph of $\mathbf{Z}_G^3$. (Note that this differs from some standard usages in that we do not require a cluster to be connected.) Each cluster A corresponds naturally to a closed subset of $\mathbf{R}^3$ which we shall denote $\mathcal{R}(A)$:

$$\mathcal{R}(A) = V(A) \cup \bigcup_{e \in E(A)} \langle e \rangle.$$

(This $\mathcal{R}(\cdot)$ notation will be extended to a larger class of objects in Sect. 3.) A cluster A is *connected* if and only if $\mathcal{R}(A)$ is a connected set (this clearly agrees with the usual definition of connected graph). A *lattice animal* is a finite connected cluster.

Let β be a lattice box. If $e \in E(\mathbf{Z}_G^3)$, then we shall say that β contains e if $\langle e \rangle \subset \beta$ (i.e., if β contains both endpoints of e). Let A be a cluster. We define $A \cap \beta$ to be the graph consisting of those edges and vertices of A that are contained in β. We define $A \setminus \beta$ to be the graph consisting of those vertices and edges of A that are not contained in β, as well as every vertex of β that is an endpoint of an edge of $A \setminus \beta$. (E.g., let A be the animal such that $\mathcal{R}(A)$ is the line segment from $(0, 0, 0)$ to $(6, 0, 0)$. Then $A \setminus [0, 2]^3$ includes the edge $\{(2, 0, 0), (3, 0, 0)\}$ and the vertex $(2, 0, 0)$, but not $(0, 0, 0)$.)

For $N \geq 0$, an *N-step walk* (or a *walk of length N*) in a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, \ldots, v_{N-1}, e_N, v_N$ in G such that the endpoints of e_i are v_{i-1} and v_i for each $i = 1, \ldots, N$. We say that this is a walk from v_0 to v_N, and we call v_0 and v_N the *endpoints* of the walk. An N-step walk is a *path* if $v_0, v_1, \ldots, v_N$ are all distinct. For $N \geq 2$, an *N-step cycle* is a graph consisting of the edges and vertices of an N-step walk in which $v_0 = v_N$, the vertices $v_1, \ldots, v_N$ are all distinct, and the N edges are all distinct (the condition on the edges is redundant unless $N = 2$). Only graphs with multiple edges have 2-step cycles. In other references, a path is sometimes called a self-avoiding walk and a cycle is sometimes called a self-avoiding polygon [23].

Let $\mathcal{P}$ be the set of all planes P in $\mathbf{R}^3$ of the form

$$P = \{(x_1, x_2, x_3) : x_k = n + 1/2\} \quad \text{where } n \in \mathbf{Z} \text{ and } k \in \{1, 2, 3\}\,.$$

A *$\mathcal{P}$-section of a set X* is a nonempty set of the form $P \cap X$ for some plane P in $\mathcal{P}$. If $e \in E(\mathbf{Z}_G^3)$ and $P \in \mathcal{P}$, then we shall say that P *touches* e (and e touches P) if $P \cap \langle e \rangle \neq \emptyset$. Observe that a given edge e touches exactly one plane P in $\mathcal{P}$, and that plane is the perpendicular bisector of $\langle e \rangle$. Similarly, we say that a $\mathcal{P}$-section $P \cap X$ touches e if $P \cap X \cap \langle e \rangle \neq \emptyset$.

When discussing the "coverage graph" in Sect. 4, we shall need the concept of a *directed graph*. A directed graph is a graph in which every edge is oriented. Thus, if Γ is a directed graph, then its edge set $E(\Gamma)$ is a set of ordered pairs $\overrightarrow{(x, y)}$ where x and y are vertices of Γ.

In a directed graph, a directed walk (or directed path or directed cycle) $v_0, e_1, v_1, \dots, v_N$ must have $e_i = \overrightarrow{(v_{i-1}, v_i)}$. In this paper we only consider directed graphs with no multiple edges (although both $\overrightarrow{(x, y)}$ and $\overrightarrow{(y, x)}$ may be edges of the same directed graph). Hence any directed walk can be described by its sequence of vertices only. The *outdegree* of a vertex x is the number of edges $\overrightarrow{(x, z)}$ in the directed graph.

2.2 Topological Concepts

We shall use topological terminology specialized for the three-dimensional context in which we work. We follow the definitions of [10]. A *ball* (respectively, a *sphere*) is a closed simplicial complex in $\mathbf{R}^3$ that is homeomorphic to $\{x \in \mathbf{R}^3 : \|x\|_2 \leq 1\}$ (respectively $\{x \in \mathbf{R}^3 : \|x\|_2 = 1\}$) where $\|\cdot\|_2$ is the Euclidean norm (a simplicial complex is a union of simplices whose pairwise intersections are faces of the simplices; see e.g. [8]). The complement of a sphere has two connected components.

Let A be a closed subset of $\mathbf{R}^3$, and let S be a sphere. We say that S *separates* A if $S \cap A = \emptyset$ and A intersects both connected components of $\mathbf{R}^3 \setminus S$.

Let $\mathcal{F}$ be the set of all finite clusters F of $\mathbf{Z}^3_G$ such that $\mathcal{R}(F)$ is not separated by a sphere. The elements of $\mathcal{F}$ are called *entangled* or *unsplittable* finite clusters. Let e_N denote the number of clusters in $\mathcal{F}$ that have exactly N edges and contain the origin. Let $\mathcal{E}_0$ be the set of subgraphs G of $\mathbf{Z}^3_G$ such that for every finite subgraph G' of G there is an entangled cluster $F \in \mathcal{F}$ such that $\mathcal{R}(G') \subset \mathcal{R}(F) \subset \mathcal{R}(G)$. Also let

$$\mathcal{E}_1 = \{G \text{ subgraph of } \mathbf{Z}^3_G : \mathcal{R}(G) \text{ is not separated by a sphere}\}.$$

See Fig. 4, as well as Holroyd [14, Fig. 3]. Grimmett and Holroyd [10] show that $\mathcal{E}_0$ and $\mathcal{E}_1$ are extremal in the sense that any collection $\mathcal{E}$ of subgraphs of $\mathbf{Z}^3_G$ that form a "measurable entanglement system" must satisfy $\mathcal{E}_0 \subseteq \mathcal{E} \subseteq \mathcal{E}_1$.

For $i \in \{0, 1\}$ and a subgraph G of $\mathbf{Z}^3_G$, we define an $\mathcal{E}_i$-component of G to be a maximal member of $\mathcal{E}_i$ that is a subgraph of G. As noted by Grimmett and Holroyd [10], the $\mathcal{E}_i$-components of G partition G.

2.3 Percolation

We consider the bond percolation model with parameter p on the lattice $\mathbf{Z}^3_G$ [9]. Each edge of $\mathbf{Z}^3_G$ is *open* with probability p, and it is *closed* otherwise. Under the product measure

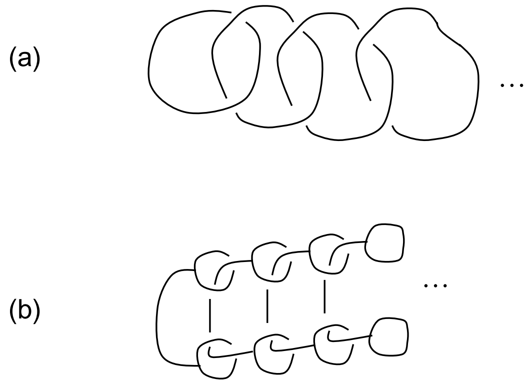


Fig. 4 Two sketches of infinite entangled graphs comprised of infinitely many finite connected components. (**a**) This graph is in $\mathcal{E}_0$ and in $\mathcal{E}_1$. (**b**) This graph is in $\mathcal{E}_1$ but not in $\mathcal{E}_0$

P_p, all edges are independent. Let W be the random subgraph of $\mathbf{Z}^3_G$ consisting of all open edges (and all vertices), and let W^* be the connected component of W containing the origin. Define $\theta(p)$ to be the probability (under P_p) that W^* is infinite. Then the standard *critical probability* p_c is defined to be the supremum of all p such that $\theta(p) = 0$. It is known that $0 < p_c < 1$, and that $P_p(|E(W^*)| \geq n)$ is an exponentially decaying function of n for each p in $(0, p_c)$ (see [9] for proofs and more properties).

For $i \in \{0, 1\}$, we write $C(\mathcal{E}_i)$ for the $\mathcal{E}_i$-component in the random graph W that contains the origin. We write $|C(\mathcal{E}_i)|$ for the number of edges in $C(\mathcal{E}_i)$. The critical probability for $\mathcal{E}_i$-entanglement is

$$p_e^i := \sup\{p : P_p(|C(\mathcal{E}_i)| = \infty) = 0\}.$$

Since $\mathcal{E}_0 \subset \mathcal{E}_1$, it follows that $C(\mathcal{E}_0) \subset C(\mathcal{E}_1)$ and $p_e^1 \leq p_e^0$.

3 Block-Clusters

In this section we introduce the block-cluster, a geometrical graph-like structure in which some vertices may be replaced by boxes. The formal definition is as follows.

A *block-cluster* D is a triple (V, E, B) where $V = V(D)$ is a set of vertices of $\mathbf{Z}^3_G$, $E = E(D)$ is a set of edges of $\mathbf{Z}^3_G$, and $B = B(D)$ is a set of lattice boxes in $\mathbf{R}^3$, such that:

(a) the lattice boxes of B are pairwise disjoint;
(b) no vertex of V is contained in a box of B;
(c) each endpoint of each edge in E either is in V or is a point of some box in B;
(d) no box of B contains an interior point of any edge of E;
(e) the sets V, E, and B are all finite, and at least one is nonempty.

Suppose that $e \in E(D)$, and let v be an endpoint of e (in $\mathbf{Z}^3_G$). Then either $v \in V(D)$ or else $v \in \beta$ for some box $\beta \in B(D)$. We shall write $v^{[D]} := v$ in the first case and $v^{[D]} := \beta$ in the second case. Let u be the other endpoint of e. Observe that if $v^{[D]}$ and $u^{[D]}$ are both boxes in $B(D)$, then the two boxes must be distinct (and hence disjoint).

If D is a block-cluster, then we can form a graph $\mathcal{G}(D)$ (an abstract graph, not necessarily embedded in $\mathbf{R}^3$) whose edge set is $E(D)$, whose vertex set is $V(D) \cup B(D)$, and such that if the edge $e \in E(D)$ has endpoints v and u in $\mathbf{Z}^3_G$, then the endpoints of e in $\mathcal{G}(D)$ are $v^{[D]}$ and $u^{[D]}$. (The idea here is that $\mathcal{G}(D)$ describes the topology of D if we shrink each box to a point. This idea is formalized in Proposition 5 below.) It is possible for $\mathcal{G}(D)$ to have multiple edges.

We now extend the notation $\mathcal{R}(\cdot)$ from Sect. 2.1. If D is a block-cluster, then let $\mathcal{R}(D)$ be the subset of $\mathbf{R}^3$ formed by the union of all boxes in $B(D)$, all vertices in $V(D)$, and all segments $\langle e \rangle$ such that e is an edge in $E(D)$. Note that $\mathcal{R}(D)$ is a closed, bounded, nonempty subset of $\mathbf{R}^3$. Since the boxes of B must be disjoint and two- or three-dimensional, the set $\mathcal{R}(D)$ determines D uniquely. Thus a block-cluster D can be represented equally well by either the triple (V, E, B) or by the set $\mathcal{R}(D)$. Observe that a lattice animal is a block-cluster in which B is empty and $\mathcal{R}(D)$ is connected. If B is a set of disjoint lattice boxes, then we shall sometimes write $\mathcal{R}(B)$ to denote the subset of $\mathbf{R}^3$ which is the union of the boxes in B (this convention treats B as the block-cluster $(\emptyset, \emptyset, B)$).

For $\epsilon > 0$ and $S \subset \mathbf{R}^3$, we define the *ϵ-neighbourhood of S* (written $N_\epsilon[S]$) to be the set of all points of $\mathbf{R}^3$ whose distance from at least one point of S is less than ϵ. The following observations are clear.

Proposition 5 *Let D be a block-cluster.*

(a) *$\mathcal{G}(D)$ is a connected graph if and only if $\mathcal{R}(D)$ is a connected set.*
(b) *Let $\epsilon \in (0, 1/2)$, and assume $\mathcal{R}(D)$ is connected. If $\mathcal{G}(D)$ has no cycles, then $N_\epsilon[\mathcal{R}(D)]$ is a ball and the boundary of $N_\epsilon[\mathcal{R}(D)]$ is a sphere.*

If $D = (V, E, B)$ and $D' = (V', E', B')$ are block-clusters, we say that D' is a sub-block-cluster of D if $V' \subset V$, $E' \subset E$, and $B' \subset B$. It is important to note that B' and B are sets of boxes, so when we write "$B' \subset B$" we mean that each box of B' is also a box of B. (If $B' \subset B$, then no box of B' is a proper subset of any box of B; in particular, if B has k boxes, then there are exactly 2^k possibilities for what B' could be.) For two block-clusters D and D', note that if D' is a sub-block-cluster of D then $\mathcal{R}(D') \subset \mathcal{R}(D)$, but that the converse is false.

A *block-cycle* is a block-cluster $D' = (V', E', B')$ such that $\mathcal{G}(D')$ is a cycle. If D is a block-cluster, then we say that D' is a block-cycle of D if D' is a block-cycle that is a sub-block-cluster of D.

4 Boxing the Loops

To help characterize and localize entanglement, we define the following transformation on block-clusters. Essentially, Ψ replaces block-cycles by minimal sets of boxes that contain them.

Algorithm: $\Psi(D)$
Let $D = (V, E, B)$ be a block-cluster. This algorithm defines another block-cluster $\Psi(D) = (V', E', B')$ as follows.

1. Let $\tilde{C}(D) = \bigcup_C \mathcal{R}(C)$, where the union is over all block-cycles C in D.
2. Let B' be the minimal collection of pairwise disjoint lattice boxes such that $\mathcal{R}(B')$ contains $\mathcal{R}(B) \cup \tilde{C}(D)$. (That is: if B^* is another collection of pairwise disjoint lattice boxes such that $\mathcal{R}(B^*)$ contains $\mathcal{R}(B) \cup \tilde{C}(D)$, then $\mathcal{R}(B') \subset \mathcal{R}(B^*)$.) We can construct B' as follows:
   2.1. Initially, let $\hat{B}_0 = \mathcal{R}(B) \cup \tilde{C}(D)$ and let $k = 0$.
   2.2. Let $\{\alpha_j\}_j$ be the collection of connected components of $\hat{B}_k$, and let $\hat{B}_{k+1} = \{\mathrm{Box}(\alpha_j)\}_j$. (Note that α_j cannot be contained in a line segment since it contains a lattice box or a block-cycle; hence $\mathrm{Box}(\alpha_j)$ is a lattice box.)
   2.3. If the boxes in $\hat{B}_{k+1}$ are all pairwise disjoint, then let $B' = \hat{B}_{k+1}$ and stop. Otherwise, increase k by one and go to Step 2.2.
3. Let V' be the set of vertices in V that are not contained in $\mathcal{R}(B')$. Let E' be the set of edges $e \in E$ such that $\langle e \rangle$ is not contained in $\mathcal{R}(B')$.

Then define $\Psi(D)$ to be the block-cluster (V', E', B'). See Fig. 5.

The following observations are immediate.

Proposition 6 *Suppose $(V', E', B') = \Psi(V, E, B)$. Then:*

(a) *$V' \subset V$ and $E' \subset E$, while $\mathcal{R}(B') \supset \mathcal{R}(B)$ and $\mathcal{R}(V', E', B') \supset \mathcal{R}(V, E, B)$;*
(b) *$\mathcal{R}(V', E', B') \subset Box(\mathcal{R}(V, E, B))$; and*
(c) *$\tilde{C}(V, E, B)$ is nonempty if and only if $B' \neq B$.*

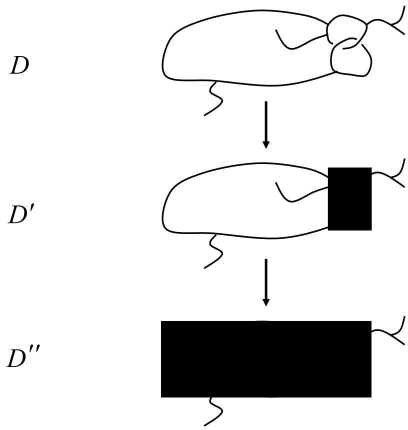


Fig. 5 Three block-clusters D, D', and D'', related by iteration of Ψ. The top block-cluster D is a lattice animal consisting of two small cycles, a long walk joining the cycles, and some short walks that have no effect on entanglement. Solid rectangles denote lattice boxes. We have $D' = \Psi(D)$, $D'' = \Psi(D')$, and $D'' = \Psi(D'')$. Also, D'' is $\Psi^{\infty}(D)$, and Block(D) consists of the black box shown in the sketch of D''

We would now like to iterate Ψ. Given a block-cluster D, define $\Psi^{(0)}(D) = D$, and $\Psi^{(i)}(D) = \Psi(\Psi^{(i-1)}(D))$ for $i \geq 1$. It follows from Proposition 6(a, b) that there exists a finite nonnegative integer $T = T(D)$ such that $\Psi^{(T)}(D) = \Psi^{(i)}(D)$ for all $i \geq T$. Accordingly, define

$$\Psi^{\infty}(D) := \Psi^{(T)}(D) \quad \text{and} \quad \mathrm{Block}(D) := B(\Psi^{\infty}(D)).$$

Observe that $\Psi^{\infty}(D)$ is a block-cluster and Block(D) is a collection of pairwise disjoint lattice boxes (the disjointness will be used frequently and tacitly in our proofs). In Fig. 5, we have $T = 2$, $D'' = \Psi^{\infty}(D)$, and Block(D) is a single box.

The following results present various basic properties of Ψ and Ψ^{∞} which will be useful later.

Proposition 7 *Let D be a block-cluster.*

(a) *The graph $\mathcal{G}(\Psi^{\infty}(D))$ has no cycles.*
(b) $\Psi^{\infty}(\Psi^{\infty}(D)) = \Psi^{\infty}(D)$.
(c) *For $0 < \epsilon < 1/2$, the ϵ-neighbourhood of each connected component of $\mathcal{R}(\Psi^{\infty}(D))$ is homeomorphic to a ball, and has boundary that is homeomorphic to a sphere.*

Proof (a) By Proposition 6(c), $\Psi^{\infty}(D)$ has no block-cycles. Therefore $\mathcal{G}(\Psi^{\infty}(D))$ has no cycles.

(b) This follows from $\Psi(\Psi^{\infty}(D)) = \Psi^{\infty}(D)$, which is a consequence of the definition of $T(D)$.

(c) This follows from part (a) and Proposition 5(b). $\square$

Lemma 8 *Assume that $D = (V, E, B)$ and $\hat{D} = (\hat{V}, \hat{E}, \hat{B})$ are block-clusters such that $\mathcal{R}(D) \subset \mathcal{R}(\hat{D})$. Then*

(a) $\mathcal{R}(\Psi(D)) \subset \mathcal{R}(\Psi(\hat{D}))$,
(b) $\mathcal{R}(\Psi^{\infty}(D)) \subset \mathcal{R}(\Psi^{\infty}(\hat{D}))$, *and*
(c) *every box of* Block(D) *is contained in some box of* Block$(\hat{D})$.

Proof Part (b) follows from repeated application of (a), and (c) is a direct consequence of (b). To prove (a), it suffices to show that $\mathcal{R}(B) \cup \tilde{C}(D) \subset \mathcal{R}(\hat{B}) \cup \tilde{C}(\hat{D})$. We know

$\mathcal{R}(B) \subset \mathcal{R}(\hat{B})$ [since $\mathcal{R}(D) \subset \mathcal{R}(\hat{D})$]. Let C_D be a block-cycle in D. Then $\mathcal{R}(C_D) \subset \mathcal{R}(D) \subset \mathcal{R}(\hat{D})$, but it still remains to show that $\mathcal{R}(C_D)$ is contained in $\mathcal{R}(\hat{B}) \cup \tilde{C}(\hat{D})$. If β is a box of $B(C_D)$, then $\beta \subset \mathcal{R}(B) \subset \mathcal{R}(\hat{B})$. To complete the proof, we must show

$$\text{If } e \text{ is an edge of } E(C_D)\text{, then } \langle e\rangle \subset \mathcal{R}(\hat{B}) \cup \tilde{C}(\hat{D}). \tag{11}$$

So assume $e \in E(C_D)$ and $\langle e\rangle \not\subset \mathcal{R}(\hat{B})$. Then $e \in \hat{E}$. Let u and v be the endpoints of e in $\mathbf{Z}^3_G$. Let D^* be the sub-block-cluster of $\hat{D}$ obtained by deleting e [i.e., $D^* = (\hat{V}, \hat{E} \setminus \{e\}, \hat{B})$]. Similarly, let C^* be the sub-block-cluster of C_D obtained by deleting e. Since $\mathcal{R}(C^*)$ is connected and $\mathcal{R}(C^*) \subset \mathcal{R}(D^*)$, the points u and v are in the same connected component of $\mathcal{R}(D^*)$. Therefore there is a path P^* in $\mathcal{G}(D^*)$ from $u^{[D^*]}$ to $v^{[D^*]}$. Then $e \cup P^*$ is a cycle in $\mathcal{G}(\hat{D})$, which corresponds to a block-cycle $\hat{C}$ in $\hat{D}$ containing e. Therefore $\langle e\rangle \subset \tilde{C}(\hat{D})$. This proves (11) and the lemma. □

Corollary 9 *Let D_0 be a block-cluster and let C_Z be a cycle of $\mathbf{Z}^3_G$ such that $\mathcal{R}(C_Z) \subset \mathcal{R}(D_0)$. Then there is a box of* $\mathrm{Block}(D_0)$ *that contains all of $\mathcal{R}(C_Z)$.*

Proof Clearly $\Psi^\infty(C_Z)$ consists of a single box. So we simply apply Lemma 8(c) with $D = C_Z$ and $\hat{D} = D_0$. □

Corollary 10 *Let A be a lattice animal and let β be a box of* $\mathrm{Block}(A)$. *Let W be a path in A with both endpoints in β. Then $\mathcal{R}(W) \subset \beta$.*

Proof We proceed by contradiction. Assume e is an edge of W that is not contained in β. Let $\tilde{W}$ be the subgraph of W containing e such that $\tilde{W}$ is a path having both endpoints in β but no other points in β. Then $\beta \cup \mathcal{R}(\tilde{W})$ determines a block-cycle and $\beta \cup \mathcal{R}(\tilde{W}) \subset \mathcal{R}(\Psi^\infty(A))$. Also, $\Psi(\beta \cup \mathcal{R}(\tilde{W}))$ is a single box. Therefore, by Lemma 8(c), $\beta \cup \mathcal{R}(\tilde{W})$ lies in a box of $\Psi^\infty(A)$, which must be β. This contradicts the assumption that e is not contained in β. □

The following lemma is key to our argument. It shows a way that the boxes of $\Psi^\infty(A)$ indicate where the cycle-like parts of A are.

Lemma 11 *Assume A is a lattice animal. Then every $\mathcal{P}$-section of every box in* $\mathrm{Block}(A)$ *touches at least two edges of A.*

Proof Let β be a box in $\mathrm{Block}(A)$, and let $P \in \mathcal{P}$. Assume that P touches at most one edge of $A \cap \beta$. We shall derive a contradiction.

Let $D = \Psi^\infty(A)$. Suppose $P = \{(x_1, x_2, x_3) : x_k = n + 1/2\}$. Then let

$$\beta_1 = \beta \cap \{(x_1, x_2, x_3) : x_k \le n\} \quad \text{and} \quad \beta_2 = \beta \cap \{(x_1, x_2, x_3) : x_k \ge n+1\}.$$

(Notice that if β is two-dimensional, then β_1 or β_2 could be a line segment, and hence not a lattice box.) Define $S \subset \mathbf{R}^3$ by

$$S = \begin{cases} (\mathcal{R}(D) \setminus \beta) \cup \beta_1 \cup \beta_2 & \text{if } P \text{ touches no edge of } A \cap \beta; \\ (\mathcal{R}(D) \setminus \beta) \cup \beta_1 \cup \beta_2 \cup \mathcal{R}(e) & \text{if } P \text{ touches the edge } e \text{ of } A \cap \beta. \end{cases}$$

Then there is a block-cluster $\tilde{D}$ such that $\mathcal{R}(\tilde{D}) = S$. (Observe that if β_i is a lattice box, then $\beta_i \in B(\tilde{D})$; otherwise, β_i is a line segment and every edge in β_i is in $E(\tilde{D})$.) Then we have $\mathcal{R}(A) \subset S = \mathcal{R}(\tilde{D})$.

The graph $\mathcal{G}(\tilde{D})$ is obtained from $\mathcal{G}(D)$ by replacing one vertex (corresponding to β) with a graph (corresponding to $\mathcal{G}(\tilde{D} \cap \beta)$) that has no cycles. Since $\mathcal{G}(D)$ has no cycles (by Proposition 7(a)), $\mathcal{G}(\tilde{D})$ has no cycles. Hence $\Psi^\infty(\tilde{D})$ has no block-cycles. Therefore $\Psi^\infty(\tilde{D}) = \tilde{D}$. Moreover, since $\mathcal{R}(A) \subset \mathcal{R}(\tilde{D})$, Lemma 8(b) implies that $\mathcal{R}(\Psi^\infty(A)) \subset \mathcal{R}(\Psi^\infty(\tilde{D}))$. This says that $\mathcal{R}(D) \subset \mathcal{R}(\tilde{D})$, which is false. This contradiction proves the lemma. □

Let $A_1, \dots, A_M$ ($M \geq 1$) be the connected components of a finite cluster L. If $\mathcal{R}(A_j) \cap \mathcal{R}(\Psi^\infty(A_i)) \neq \emptyset$ (for given $i \neq j$), then we say that A_i *covers* A_j and we write $A_i \searrow A_j$. That is, A_i covers A_j if some box of $\mathrm{Block}(A_i)$ touches A_j. The *coverage graph of* L, denoted $\mathrm{Cover}(L)$, is the directed graph whose vertices are $1, \dots, M$ and whose edge set is

$$\{\overrightarrow{(i, j)} : A_i \searrow A_j\}.$$

Proposition 12 *Let L be an entangled finite cluster (i.e., $L \in \mathcal{F}$), and assume that L is not connected. Then*

(a) *no vertex of* $\mathrm{Cover}(L)$ *has outdegree* 0, *and*
(b) $\mathrm{Cover}(L)$ *contains a directed cycle (of length* 2 *or more).*

The proof of part (b) uses the following elementary lemma about directed graphs.

Lemma 13 *Let Γ be a finite directed graph with vertex set V. Assume no vertex of V has outdegree* 0. *Then Γ has a directed cycle.*

Proof We shall inductively define a sequence of vertices $\{w_i\}$ of V as follows. Let $w_0 \in V$. Since the outdegree of w_0 is positive, there is a vertex w_1 such that $\overrightarrow{(w_0, w_1)}$ is an edge of Γ. Continuing in this way, suppose we have constructed a directed walk with vertices $w_0, w_1, \dots, w_i$. Let w_{i+1} be a vertex such that $\overrightarrow{(w_i, w_{i+1})}$ is an edge of Γ. Since V is finite, there must be a smallest positive integer j such that $w_j = w_i$ for some $i \in \{0, \dots, j-1\}$. Then $(w_i, \dots, w_j)$ is a directed cycle in Γ. □

Proof of Proposition 12 Assume vertex 1 has outdegree 0, i.e. $\mathcal{R}(\Psi^\infty(A_1)) \cap A_j = \emptyset$ for every $j \geq 2$. Thus we see that for $\epsilon \in (0, 1/2)$, the boundary of the ϵ-neighbourhood of $\mathcal{R}(\Psi^\infty(A_1))$ separates A_1 from $\bigcup_{j=2}^M A_j$. And by Proposition 7(c), this boundary is a sphere. This contradicts the fact that $L \in \mathcal{F}$. This proves (a). Part (b) follows directly from part (a) and Lemma 13. □

5 Connecting the Components

Our goal in this section is to show that one can add a controlled number of edges to any entangled cluster so as to join up all of the components. We begin with an easy result (Proposition 14 and its ensuing remark) that indicates the kind of thing we want to do. However, we shall soon see that the stronger version that we need (Proposition 15) requires a much more elaborate proof.

Proposition 14 *Let A_1 and A_2 be two lattice animals and assume that $A_1 \searrow A_2$. Then there is a box $\beta_1 \in \mathrm{Block}(A_1)$ and a path π in $\mathbf{Z}_G^3$ from a vertex of A_1 to a vertex of A_2 such that $\mathcal{R}(\pi) \subset \beta_1$ and $|E(\pi)| \leq \mathrm{diam}(\beta_1)$.*

Proof Since $A_1 \searrow A_2$, there is a block β_1 of Block(A_1) and a vertex x_2 of A_2 such that $x_2 \in \beta_1$. Choose any vertex x_1 of $A_1 \cap \beta_1$, and let π be a path of shortest length in $\mathbf{Z}_G^3$ from x_1 to x_2. Then $\mathcal{R}(\pi) \subset \beta_1$. Since π was chosen to have shortest length, it follows that $|E(\pi)| \leq \mathrm{diam}(\beta_1)$. □

Remark In the context of Proposition 14, the graph $A_1 \cup A_2 \cup \pi$ is connected. Thus if $A_1 \cup A_2$ is an entangled cluster with A_1 and A_2 connected and disjoint, then $A_1 \searrow A_2$, so we can join up A_1 and A_2 by adding at most $|E(\pi)|$ edges to the cluster for some path π with $|E(\pi)| \leq |E(A_1)|$ (in fact $|E(\pi)| \leq |E(A_1)|/2$ by Lemma 11).

We now introduce some terminology that will be important in the subsequent development.

A *Green/Red graph* is a graph in which each edge is coloured either Green or Red. (We shall only use this when the graph is a cluster.) For a Green/Red graph L, let Green(L) [respectively, Red(L)] denote the set of Green [respectively, Red] edges of L.

We say that a Green/Red animal A has the GG Property if

(GG1) every edge of $\Psi^\infty(A) \setminus \mathrm{Block}(A)$ is Green in A [i.e. $E(\Psi^\infty(A)) \subset \mathrm{Green}(A)$], and
(GG2) every $\mathcal{P}$-section of every box in Block(A) touches at least two Green edges of A.

We say that a Green/Red cluster has the GG Property if every connected component of the cluster has the GG Property.

Consider an entangled cluster L in which every edge is Green. Then Lemma 11 tells us that L has the GG Property. The following result says that we can add new Green edges to L to join up some components, while changing two pre-existing Green edges to Red for each new Green edge, in such a way as to preserve the GG Property. We note that Proposition 14 is not sufficient to guarantee preservation of the (GG2) property.

Proposition 15 *Suppose L is a Green/Red cluster that has the GG Property. Also suppose that some of the connected components of L, say $A_1, \ldots, A_q$ (with $q > 1$) form a directed cycle in Cover(L), i.e. $A_{i-1} \searrow A_i$ for $i = 1, \ldots, q$ (where we define A_0 to be A_q). Then there is a Green/Red cluster L' such that*

(a) *L is a subgraph of L' (with edge colours in L' not necessarily the same as in L);*
(b) *all edges of $E(L') \setminus E(L)$ are Green;*
(c) $\mathrm{Red}(L) \subset \mathrm{Red}(L')$;
(d) $|\mathrm{Red}(L') \setminus \mathrm{Red}(L)| = 2|E(L') \setminus E(L)|$;
(e) *$A_1, \ldots, A_q$ are all in the same connected component of L', and each of the other connected components of L' equals a connected component of L (with the same edge colours); and*
(f) *L' has the GG Property.*

It follows that $\mathrm{Red}(L') \subset E(L)$ [*by* (b)], *that L' has fewer connected components than L* [*by* (e)], *and*

$$|\mathrm{Red}(L)| + 2|\mathrm{Green}(L)| = |\mathrm{Red}(L')| + 2|\mathrm{Green}(L')|. \tag{12}$$

Before proving this result, we present the following corollary. It says that by repeating the process of this Proposition, the connected components of an entangled cluster with N edges can all be joined up by adding at most N additional edges. This leads directly to the exponential upper bound on the number of entangled clusters (Theorem 1, which we shall prove in Sect. 6.1).

Corollary 16 *Let L_0 be an entangled cluster with N edges. Then L_0 is a subgraph of a lattice animal that has at most $2N$ edges.*

Proof This is trivial if L_0 is connected, so assume L_0 is not connected. Let every edge of L_0 be coloured Green. Then Lemma 11 tells us that L_0 has the GG Property. We now inductively define a (finite) sequence of finite entangled Green/Red clusters $\{L_i\}$ having the GG Property. Assume L_i is a finite entangled cluster with the GG Property, and is not connected. Then Proposition 12 shows that Cover(L_i) has a directed cycle. Let L_{i+1} be the Green/Red cluster L' obtained by applying Proposition 15 with $L = L_i$. Then L_{i+1} is finite and has the GG Property. Since L_{i+1} is obtained from the entangled cluster L_i by adding edges but not creating new components, L_{i+1} must also be entangled. Since L_{i+1} has fewer connected components than L_i, we eventually obtain a Green/Red cluster L_I with only one component. From Red$(L_0) = \emptyset$ and (12), we see that

$$2N = |\mathrm{Red}(L_0)| + 2|\mathrm{Green}(L_0)| = |\mathrm{Red}(L_I)| + 2|\mathrm{Green}(L_I)| \geq |E(L_I)|,$$

so L_I is the desired lattice animal. □

The following lemma will be used a few times in the proof of Proposition 15.

Lemma 17 *Assume $2 \leq q \leq s$. Let $A_1, \ldots, A_s$ be pairwise disjoint Green/Red lattice animals that each has the GG Property. Let A' be a Green/Red animal containing every animal A_i (the edges need not have the same colour in A' as in $\cup A_i$). Let β_i be a box of* Block(A_i) *for each $i = 1, \ldots, q$, and let $\beta_i = \emptyset$ for $i = q+1, \ldots, s$. Let $\beta_\cup = \bigcup_{j=1}^{q} \beta_j$. Assume*

(i) *every edge of A' that is not contained in $\beta_\cup$ is an edge of A_i for some $i \leq s$, and*
(ii) *every edge of $A_i \setminus \beta_i$ has the same colour in A' as it has in A_i.*

Then

(a) *A' has the (GG1) property, and*
(b) *If $P \in \mathcal{P}$, $\beta^0 \in$ Block(A'), $P \cap \beta^0 \neq \emptyset$, and $P \cap \beta^0 \cap \beta_\cup = \emptyset$, then $P \cap \beta^0$ touches at least two Green edges of A'.*

Proof (a): Let e be an edge of $E(\Psi^\infty(A'))$. In particular, $\langle e \rangle \not\subset \mathcal{R}(\mathrm{Block}(A'))$, so e is not in any of $\beta_1, \ldots, \beta_q$. Therefore e must be an edge of A_i for some $i \leq s$, and e must have the same colour in A' as in A_i. By Lemma 8(c), e cannot be in any box of Block(A_i). Therefore e is Green in A_i [since (GG1) holds for A_i], and hence e is Green in A'. We conclude that (GG1) holds for A'.

(b): On the one hand, suppose $P \cap \beta^0$ touches a box β^* of Block(A_k) for some $k \leq s$. By Lemma 8(c), $\beta^* \subset \beta^0$. Since $P \cap \beta^0 \cap \beta_\cup = \emptyset$, we know $\beta^* \neq \beta_k$, so $\beta^* \cap \beta_k = \emptyset$. Therefore, by (GG2) for A_k, $P \cap \beta^0$ touches at least two Green edges of $A_k \cap \beta^*$, hence of $(A_k \setminus \beta_k) \cap \beta^0$, hence of $A' \cap \beta^0$.

On the other hand, suppose that for every $k \leq s$, $P \cap \beta^0$ touches no box of Block(A_k). Lemma 11 shows that $P \cap \beta^0$ touches at least two edges of A', say f_1 and f_2. We know that $f_1 \not\subset \beta_\cup$, so f_1 is in A_k for some $k \leq s$. By (GG1), f_1 is Green in A_k, so f_1 is Green in A'. Similarly, f_2 is Green in A'. □

Proof of Proposition 15 First we show that (a)–(d) imply (12). Let $R\Delta = \mathrm{Red}(L') \setminus \mathrm{Red}(L)$. Then $R\Delta \subset \mathrm{Green}(L)$ [by (b)], and

$$|E(L') \setminus E(L)| = \frac{1}{2}|R\Delta| = \frac{1}{2}\left(|\mathrm{Red}(L')| - |\mathrm{Red}(L)|\right) \quad \text{[by (d) and (c)].}$$

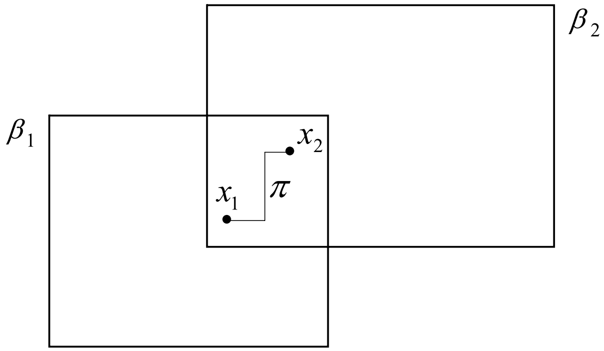


Fig. 6 Sketch of Subcase I(a). The boxes β_1 and β_2 are shown by outlined rectangles

Then $\mathrm{Green}(L') = (E(L') \setminus E(L)) \cup (\mathrm{Green}(L) \setminus R\Delta)$ and

$$|\mathrm{Green}(L')| = |E(L') \setminus E(L)| + |\mathrm{Green}(L)| - |R\Delta|,$$

from which we obtain (12).

For the rest of the proof, we shall consider the cases $q = 2$ and $q > 2$ separately.

Case I: $q = 2$. Since $A_1 \searrow A_2 \searrow A_1$, there exists, for $i = 1, 2$, a vertex x_i in $V(A_i)$ and a box β_i in $\mathrm{Block}(A_i)$ such that $x_1 \in \beta_2$ and $x_2 \in \beta_1$. Case I breaks into two subcases: either $x_1 \in \beta_1$ and $x_2 \in \beta_2$ (Subcase I(a)) or not (Subcase I(b)).

Subcase I(a): $x_1 \in \beta_1$ and $x_2 \in \beta_2$: Then x_1 and x_2 are both in $\beta_1 \cap \beta_2$. Let π be a shortest path in $\mathbf{Z}^3_G$ that joins x_1 to x_2. Then π lies in $\beta_1 \cap \beta_2$ (which is either a lattice box or a line segment). (See Fig. 6.) Let $e_1, \ldots, e_w$ be the edges of π that are not already in L. For each $k = 1, \ldots, w$, let P_k be the plane of $\mathcal{P}$ that touches e_k (i.e., the perpendicular bisector of $\langle e_k \rangle$). For $i = 1, 2$, since $e_k \subset \beta_1 \cap \beta_2$, we see that $P_k \cap \beta_i \neq \emptyset$; hence $P_k \cap \beta_i$ touches at least two Green edges of $A_i \cap \beta_i$, say $e_{ik,R}$ and $e_{ik,G}$ (since A_i has the GG Property). Observe that the edges $e_{ik,c}$ ($i = 1, 2$; $k = 1, \ldots, w$; $c = R, G$) are all distinct (to see this, use the disjointness of A_1 and A_2 as well as the fact that the planes $P_1, \ldots, P_w$ are all distinct, because π is a shortest path; note that each edge of $\mathbf{Z}^3_G$ touches only one plane of $\mathcal{P}$).

Let L' be the Green/Red cluster obtained from L by adding the Green edges $e_1, \ldots, e_w$ (and their endpoints), and changing the colour of each $e_{ik,R}$ from Green to Red. It is clear that properties (a), (b), (c), and (e) hold for L'. For (d), observe that $|E(L') \setminus E(L)| = w$ and $|\mathrm{Red}(L')| = |\mathrm{Red}(L)| + 2w$. It remains to show that (f) holds.

Let A' be the component of L' containing π. Let s be the number of components of L that π touches. Clearly $s \geq 2$, since π touches A_1 and A_2; let us denote the other components (if any) that π touches by $A_3, \ldots, A_s$. Then A' is the union of $A_1, \ldots, A_s$, and π. Since the other components of L' are unchanged from L, it suffices to show that A' has the GG Property. Lemma 17(a) implies that (GG1) holds for A'.

It remains to show that (GG2) holds for A'. Consider a plane P in $\mathcal{P}$ that intersects a box β^0 of $\mathrm{Block}(A')$. We must show

(*) $P \cap \beta^0$ touches at least two Green edges of A'.

There are two possibilities to consider:

(K1): $P \cap \beta^0 \cap (\beta_1 \cup \beta_2) = \emptyset$;
(K2): $P \cap \beta^0 \cap (\beta_1 \cup \beta_2) \neq \emptyset$.

In Case (K1), Lemma 17(b) shows that (*) holds for A'. So assume (K2) holds. Then $\beta_1 \cup \beta_2 \subset \beta^0$ by Lemma 8(c). If P touches e_k (of π) for some k, then e_k and $e_{1k,G}$ are two Green edges of $A' \cap \beta_1$ (hence of $A' \cap \beta^0$) touched by P, and (*) would follow. So suppose P touches none of the e_k's. Then the edges of L' touched by P have the same colour as they do in L. Choose $i \in \{1, 2\}$ such that P intersects β_i. Since A_i had the GG Property, we know that P touches at least two Green edges of $A_i \cap \beta_i$. But these two edges are also Green edges of A' and they are both in β^0. Hence (*) holds when (K2) does. We conclude that (GG2) holds for A' in Case (K2).

This concludes the proof of Subcase I(a).

Subcase I(b): $x_1 \notin \beta_1$ or $x_2 \notin \beta_2$ or both: Without loss of generality, assume $x_2 \notin \beta_2$. Since A_2 is connected, there is a path W_2 in A_2 from x_2 to a point y_2 on the boundary of β_2, such that y_2 is the only vertex of W_2 in β_2. Let π be a shortest path in $\mathbf{Z}_G^3$ from x_1 to y_2. Then π is contained in β_2.

Let $e_1, \ldots, e_w$ be the edges of π that are not already in L. For each $k = 1, \ldots, w$, let P_k be the plane of $\mathcal{P}$ that touches e_k. Since $e_k \in \beta_2$ and A_2 has the GG Property, $P_k \cap \beta_2$ touches at least two Green edges of $A_2 \cap \beta_2$, say $e_{2k,1}$ and $e_{2k,2}$. As in Subcase I(a), these $2w$ edges $e_{2k,i}$ are all distinct since the w planes P_k are distinct.

Let L' be the Green/Red cluster obtained from L by adding the Green edges $e_1, \ldots, e_w$ (and their endpoints), and changing the colour of each $e_{2k,i}$ from Green to Red. As in Subcase I(a), we see that properties (a) through (e) hold for L'. It remains to prove (f).

Let A' be the component of L' containing π (and hence A_1, A_2, and perhaps other components $A_3, \ldots, A_s$ of L, as in Subcase I(a)). As in Subcase I(a), it suffices to show that A' has the GG Property. Lemma 17(a) shows that Property (GG1) holds for A'. We now turn to (GG2).

By Lemma 8(c), there is a box β' of Block(A') that contains β_2. We shall prove that β' also contains β_1. This is clear if $\beta_1 \cap \beta_2 \neq \emptyset$, so assume $\beta_1 \cap \beta_2 = \emptyset$. In particular, $x_1 \notin \beta_1$. Let W_1 be a path in A_1 from x_1 to a point of β_1. For $i = 1, 2$, let $\tilde{W}_i$ be a sub-path of W_i having one endpoint in β_1, having the other endpoint in β_2, and having no other vertices in $\beta_1 \cup \beta_2$. Also, for $j = 1, 2$, there exist paths $\pi[j]$ in $\mathbf{Z}_G^3 \cap \beta_j$ from $\tilde{W}_1 \cap \beta_j$ to $\tilde{W}_2 \cap \beta_j$. Let $C_Z = \tilde{W}_1 \cup \pi[1] \cup \tilde{W}_2 \cup \pi[2]$. Then C_Z is a cycle in $\mathbf{Z}_G^3$ such that

$$\mathcal{R}(C_Z) \subset \Psi^{\infty}(A_1) \cup \Psi^{\infty}(A_2) \subset \Psi^{(\infty)}(A') \quad \text{(by Lemma 8(c))}.$$

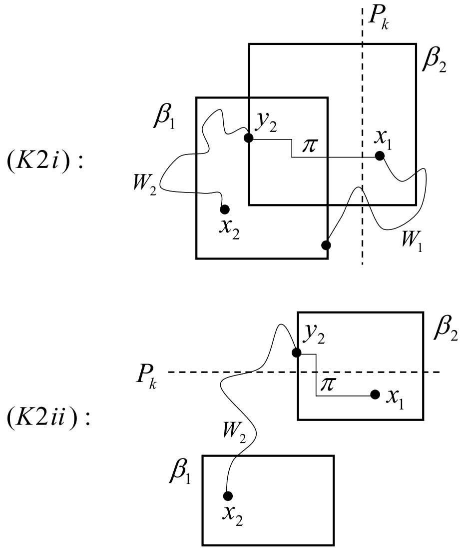
By Corollary 9, $\mathcal{R}(C_Z)$ is contained in a box β'' of Block(A'). But $\mathcal{R}(C_Z) \cap \beta' \neq \emptyset$, so $\beta'' = \beta'$. We conclude that $\beta_1 \subset \beta'$.

As in Subcase I(a), consider a plane P in $\mathcal{P}$ that intersects a box β^0 of Block(A'). We again consider the two possibilities (K1) and (K2) and try to show that (*) holds. Note that the colours outside β_2 are the same in L' as in L. If (K2) holds, then $\beta^0 = \beta'$. The arguments for (K1) and (K2) from I(a) all hold for I(b), *except* for the situation in (K2) where P touches e_k for some k. To complete Subcase I(b), then, we need only to consider this situation.

Consider the section of β' by P_k, the plane of $\mathcal{P}$ that touches e_k. We claim that $P_k \cap \beta'$ touches a Green edge of $A' \cap \beta'$ besides e_k. On the one hand, if $P_k \cap \beta_1 \neq \emptyset$, then this section must touch two Green edges of $A_1 \cap \beta_1$ (neither is an $e_{2k,i}$ because $e_{2k,i}$ is in A_2), and $A_1 \cap \beta_1 \subset A' \cap \beta'$, which proves the claim. On the other hand, if $P_k \cap \beta_1 = \emptyset$, then β_1 lies on one side of the plane P_k, and P_k separates β_1 either from x_1 or from y_2 (notice that x_1 and y_2 are on opposite sides of P_k, by definition of π and P_k). We consider these two possibilities separately (see Fig. 7).

(K2i): If P_k separates β_1 from x_1, then there is a path W_1 in A_1 from x_1 to the boundary of β_1. So P_k touches an edge e of W_1. By Corollary 10 (with $A = A'$ and $\beta = \beta'$), $\mathcal{R}(W_1) \subset \beta'$. In particular, $e \subset \beta'$. Notice that e, like every edge of A_1, has the same colour in A' as in A_1. On the one hand, if $\langle e \rangle \not\subset \mathcal{R}(\text{Block}(A_1))$, then e is Green in

Fig. 7 Sketch of the two possibilities (K2i) and (K2ii) in Subcase I(b). The boxes β_1 and β_2 are shown by outlined rectangles; they may or may not be disjoint



L [by (GG1)] and hence also Green in L', so the claim holds. On the other hand, if $\langle e\rangle \subset \mathcal{R}(\text{Block}(A_1))$, then let β_e be the box in Block(A_1) containing e. By (GG2) for A_1, $P_k \cap \beta_e$ touches at least two Green edges f_1 and f_2 of A_1. We know $\beta_e \subset \beta'$ (since $\langle e\rangle \subset \beta'$). Therefore f_1 is in $A' \cap \beta'$ and is touched by P_k, which proves the claim.

(K2ii): If P_k separates β_1 from y_2, then P_k touches an edge e of W_2. By definition of W_2, $\langle e\rangle \not\subset \beta_2$. By Corollary 10, $\mathcal{R}(W_2) \subset \beta'$, so $\langle e\rangle \subset \beta'$. On the one hand, if $\langle e\rangle \not\subset \mathcal{R}(\text{Block}(A_2))$, then e is Green in L [by (GG1)] and hence also Green in L' [since $\langle e\rangle \not\subset \beta_2$], so the claim holds. On the other hand, if $\langle e\rangle \subset \mathcal{R}(\text{Block}(A_2))$, then let β_e be the box in Block(A_2) containing e. Then $\beta_e \cap \beta_2 = \emptyset$ (since $\langle e\rangle \not\subset \beta_2$). By (GG2) for A_2, $P_k \cap \beta_e$ touches at least two Green edges f_1 and f_2 of A_2, and f_1 and f_2 are still Green in L'. We know $\beta_e \subset \beta'$ (since $\langle e\rangle \subset \beta'$). Therefore f_1 is in Green$(A') \cap \beta'$ and is touched by P_k, which proves the claim.

The claim has thus been shown to hold in all possible situations. This concludes the proof of Subcase I(b), and hence of Case I.

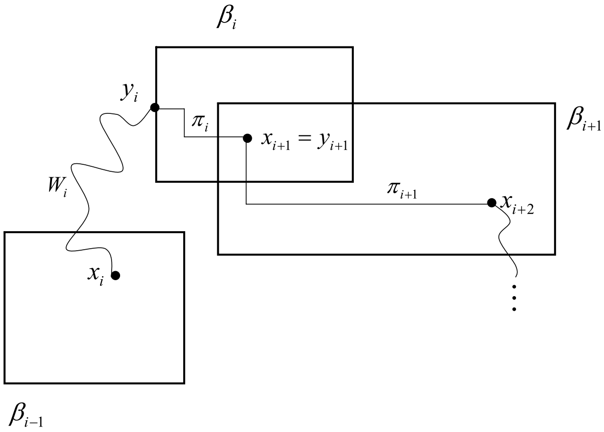
Case II: $q > 2$. Recall $A_1 \searrow A_2 \searrow \cdots \searrow A_q \searrow A_1$. Hence for each $i = 1, \ldots, q$, there exists a box β_i in Block(A_i) and a vertex x_i of A_i such that $x_i \in \beta_{i-1}$ (we write $\beta_0 = \beta_q$, etc.).

For each $i = 1, \ldots, q$, let y_i be a vertex of $A_i \cap \beta_i$ and let W_i be a path in A_i from x_i to y_i such that the following hold:

- If $x_i \in \beta_i$, then $y_i = x_i$ and W_i is the trivial path consisting of the single vertex x_i;
- If $x_i \notin \beta_i$, then y_i is a point on the boundary of β_i, and W_i contains no point of β_i besides y_i.

Next, let π_i be a shortest path in $\mathbf{Z}^3_G$ from y_i to x_{i+1} (See Fig. 8.) Then π_i is contained in β_i (since $y_i, x_{i+1} \in \beta_i$). For each $i = 1, \ldots, q$ and each edge e in π_i, let $P[e]$ be the plane in $\mathcal{P}$ that touches e. Since A_i has the GG Property, we can specify two Green edges $f_{i,1}[e]$ and $f_{i,2}[e]$ of A_i that touch $\beta_i \cap P[e]$. Observe that all of these $f_{\cdot,\cdot}[\cdot]$ edges are distinct.

Fig. 8 Sketch of Case II



Let Γ be the sequence of edges and vertices in $W_1, \pi_1, W_2, \pi_2, \ldots, W_q, \pi_q$. Then Γ is a walk in $\mathbf{Z}^3_G$ from x_1 to x_1, containing x_1, y_1, x_2, $y_2, \ldots, x_q, y_q, x_1$ in this order. (This walk will be contained in L', which we define next.)

We define the new Green/Red cluster L' by the following algorithm:

1. Let $L[0] = L$ and $i = 1$.
2. Let $w(i)$ be the number of edges of π_i that are not in $L[i-1]$. Call these edges $e_{i,k}$ $(k = 1, \ldots, w(i))$.
3. Define the Green/Red cluster $L[i]$ to be the union of $L[i-1]$ and the edges $e_{i,k}$ $(k = 1, \ldots, w(i))$, with every $e_{i,k}$ coloured Green, and with the edges $f_{i,m}[e_{i,k}]$ $(k = 1, \ldots, w(i), m = 1, 2)$ changed from Green to Red.
4. If $i < q$, then increase i by one and go to Step 2. Otherwise, set L' to be $L[q]$, and stop.

It is clear that properties (a) through (e) hold for L', so we only need to prove (f). Let A' be the component of L' containing π_1 (and hence π_i and A_i for $i = 1, \ldots, q$). As in Subcase I(a), it suffices to show that A' has the GG Property. Lemma 17(a) shows that (GG1) holds for A', so it remains to show that (GG2) holds for A'.

Consider a plane P in $\mathcal{P}$ that intersects a box β^0 of Block(A'). There are exactly two possibilities to consider (by Lemma 8(c) and the disjointness of boxes in Block(A')): either

(KK1): $(P \cap \beta^0) \cap (\bigcup_{j=1}^q \beta_j) = \emptyset$, or
(KK2): For some $i \in \{1, \ldots, q\}$, $\beta_i \subset \beta^0$ and $P \cap \beta_i \neq \emptyset$.

If (KK1) holds, then Lemma 17(b) tells us that $P \cap \beta^0$ touches at least two Green edges of A'. So assume (KK2) holds for i. If P touches none of the edges $e_{i,k}$ $(1 \le k \le w(i))$ from $L[i] \setminus L[i-1]$ of the above algorithm, then the situation is the same as in Case I [that is: since (GG2) holds for A_i, we know that $P \cap \beta_i$ touches at least two Green edges of A_i, and these edges do not become Red in A', so we are done]. So suppose that P touches one of the edges $e_{i,k}$ $(1 \le k \le w(i))$. Then $e_{i,k}$ is Green in A', so to verify (GG2), we need to show

(**) There exists an edge f in Green$(A') \setminus \{e_{i,k}\}$ that touches $P \cap \beta^0$.

We shall consider two subcases, according as to whether or not $e_{i,k}$ occurs more than once in the walk Γ.

First suppose that $e_{i,k}$ occurs more than once in Γ. The edge $e_{i,k}$ is not in L, so it cannot be in $\bigcup_{j=1}^q W_j$, and so it must be in π_j for some $j \neq i$. Then $\langle e_{i,k} \rangle \subset \beta_j$, so $\beta_i \cap \beta_j \neq \emptyset$.

Therefore $\beta_j \subset \beta^0$. Also, $P \cap \beta_j \neq \emptyset$ (since $P \cap \beta_j$ touches $e_{i,k}$). Let $f = f_{j,1}[e_{i,k}]$. Then f is a Green edge of A_j that touches $P \cap \beta_j$, and hence touches $P \cap \beta^0$. Since the edge $e_{i,k}$ is not in $L[i-1]$ (by Step 2 of the algorithm), we conclude that $j > i$ and $e_{i,k} \in E(L[i]) \subset E(L[j-1])$. Thus $e_{i,k}$ is not in the set $\{e_{j,m} : 1 \le m \le w(j)\}$. Therefore $P \cap \langle e_{j,m} \rangle = \emptyset$ for all $m = 1, \ldots, w(j)$, and so f did not turn Red when the algorithm defined $L[j]$. Thus we conclude that (**) holds when $e_{i,k}$ occurs more than once in Γ.

Now suppose that $e_{i,k}$ occurs only once in Γ. Then there is a cycle C_e in Γ that contains $e_{i,k}$. Therefore there exists an edge $\tilde{e}$ of $C_e \setminus e_{i,k}$ that touches P. Since $e_{i,k} \subset \beta_i \subset \beta^0$, Corollary 10 (with $A = A'$, $\beta = \beta^0$, and $W = C_e \setminus e_{i,k}$) implies that $\mathcal{R}(C_e) \subset \beta^0$. In particular, $\langle \tilde{e} \rangle \subset \beta^0$. We consider two subcases: either

(i) $\tilde{e}$ is in some W_j, or
(ii) $\tilde{e}$ is in some π_j $(j \neq i)$.

(i): By definition of W_j, $\langle \tilde{e} \rangle \not\subset \beta_j$. Suppose first that $\langle \tilde{e} \rangle \not\subset \mathcal{R}(\text{Block}(A_j))$. Then $\tilde{e} \in \text{Green}(A_j)$ [by (GG1) for A_j]. Since $\langle \tilde{e} \rangle \not\subset \beta_j$, we see that $\tilde{e}$ has the same colour in L' as in L. Therefore (**) holds with $f = \tilde{e}$ if $\langle \tilde{e} \rangle \not\subset \mathcal{R}(\text{Block}(A_j))$. So suppose now that $\tilde{e}$ is contained in a box $\tilde{\beta}$ of Block(A_j). By (GG2) for A_j, there is an edge f of Green(A_j) touching $P \cap \tilde{\beta}$. Since $\langle \tilde{e} \rangle \not\subset \beta_j$, we see that $\tilde{\beta} \cap \beta_j = \emptyset$, so f has the same colour in A' as in L. Finally, $\tilde{\beta} \subset \beta^0$ [because $\langle \tilde{e} \rangle \subset \beta^0$], so f touches $P \cap \beta^0$. Therefore we conclude that (**) holds in subcase (i).

(ii): First, suppose that $\tilde{e}$ equals $e_{j,m}$ for some m $(1 \le m \le w(j))$. Then $\tilde{e}$ is Green in L' and (**) holds with $f = \tilde{e}$. Next, suppose that there is no $e_{j,m}$ that equals $\tilde{e}$. Let $f = f_{j,1}[\tilde{e}]$. Then $f \subset \beta_j \subset \beta^0$ [since $\langle \tilde{e} \rangle \subset \beta_j$ and $\langle \tilde{e} \rangle \subset \beta^0$]. Therefore $P \cap \beta^0$ touches f. The edge f is Green in A_j, and it is not changed to Red when the algorithm defined $L[j]$ [because $\tilde{e}$ is not one of the $e_{j,m}$'s]. Therefore (**) holds in subcase (ii).

This completes the proof that (GG2) holds when (KK2) holds. It also completes the proof of Case II, and the Proposition. □

6 Proofs of the Main Theorems

6.1 Exponential Growth

We begin by introducing the *concatenation* of two clusters, and explain how it implies the existence (but not necessarily finiteness) of the limit $\lambda_e := \lim_{N\to\infty} e_N^{1/N}$. The argument is essentially the same as the standard argument for $\lambda := \lim_{N\to\infty} a_N^{1/N}$ [19, 20].

Definition 18 (a) For two distinct points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ of $\mathcal{R}^3$, we say that x *is lexicographically larger than* y if $x_I > y_I$ where $I = \min\{i : x_i \neq y_i\}$.

(b) Let H be a cluster and let $v \in \mathbf{Z}^3$. Then $H + v$ is the cluster obtained from translating every vertex and edge by the vector v.

(c) Let H_1 and H_2 be finite clusters. The *concatenation* of H_1 with H_2 is the graph $H_1 \circ H_2$ defined as follows. Let x be the lexicographically largest vertex of H_1, and let y be the lexicographically smallest vertex of H_2. Then $H_1 \circ H_2 := H_1 \cup (H_2 + x - y)$.

Observe that in part (c), $H_1 \cap (H_2 + x - y)$ is the one-vertex graph x. Also, if H_1 has n_1 edges and H_2 has n_2 edges, then $H_1 \circ H_2$ has $n_1 + n_2$ edges. Finally, if H_1 and H_2 are both connected (respectively, entangled) clusters, then $H_1 \circ H_2$ is also connected (respectively, entangled).

The following lemma, about "supermultiplicative" sequences, is equivalent to the well-known lemma about "subadditive" sequences (see for example Lemma 1.2.2 of [23]).

Lemma 19 *Let $u_1, u_2, \ldots$ be a sequence of positive real numbers such that*

$$u_i u_j \le u_{i+j} \quad \textit{for all } i, j \ge 1.$$

Let $L = \sup_{n\ge 1} u_n^{1/n}$ (possibly equal to $+\infty$). Then $\lim_{n\to\infty} u_n^{1/n}$ exists and equals L.

We are now able to prove Theorem 1.

Proof of Theorem 1: First we prove that the limit λ_e [defined in (4)] exists. For each positive integer n, let ENT_n be the collection of entangled clusters containing the origin and having exactly n edges. Fix positive integers m and n. Consider the map $\chi : ENT_m \times ENT_n \to ENT_{m+n}$ defined by $\chi(H_1, H_2) = H_1 \circ H_2$. We claim that H is at most $2n$-to-one. To see this, suppose $H = \chi(H_1', H_2')$. Let $v(1), v(2), \ldots, v(k)$ be the vertices of H in lexicographic order, and for $1 \le i \le j \le k$ let $H[i; j]$ be the subgraph of H induced by the vertices $\{v(i), \ldots, v(j)\}$. Then it is not hard to see that there is a unique ℓ such that $H[1, \ell]$ has m edges and $H[\ell, k]$ has n edges; indeed, $H[1, \ell] = H_1'$ and $H[\ell, k]$ is a translation of H_2'. Then H_2' must equal $H[\ell, k] - v$ for some vertex v of $H[\ell, k]$. Since the number of vertices of v is at most $2n$, this proves the claim.

The claim of the preceding paragraph implies that for all $M, N \ge 1$, we have $e_{M+N} \ge e_M e_N/(2\min\{M, N\})$ (take $n = \min\{M, N\}$ and $m = \max\{M, N\}$). Hence

$$\frac{e_{N+M}}{N+M} \ge \frac{e_M e_N}{4MN} \quad \text{for all } M, N \ge 1.$$

The existence of $\lim_{N\to\infty} e_N^{1/N}$ (perhaps $+\infty$) now follows from Lemma 19 with $u_i = e_i/(4i)$.

Corollary 16 says that every entangled cluster L with $|E(L)| = N$ is contained in a lattice animal A having at most $2N$ edges. By attaching an arbitrary connected set of edges to A if necessary, we can assume that A has exactly $2N$ edges. Observe that the same A can arise from many different clusters. Given A, the number of possible choices of L that could produce A is at most $\binom{2N}{N}$, which is less than 4^N. The bound (3) follows, as does the inequality $\lambda_e \le 4\lambda^2$.

Since every lattice animal is an entangled cluster, it is obvious that $\lambda \le \lambda_e$. It remains to prove that this inequality is strict. This is a consequence of the pattern Theorem 2.1 of [22]. In the terminology of that paper, a "cluster of size n" is our entangled cluster with n edges, and the weight of every cluster is 1. Theorem 2.1 of [22] requires Cluster Axioms (CA1), (CA2), and (CA4); the first two are obvious, and (CA4) may be checked as in Proposition 3.1 of that paper. We use the following pattern $P = (P_1, P_2)$ in Theorem 2.1 of [22]: the "occupied" set P_1 consists of $C_A \cup C_B$, where C_A is the 8-step cycle that forms the boundary of the square $[-1, 1] \times [-1, 1] \times \{0\}$, and C_B is any cycle that contains the origin and is disjoint from C_A (hence P_1 is entangled); and the "vacant" set P_2 is the set of all edges of $\mathbf{Z}_G^3$ that have exactly one endpoint in C_A. Thus any cluster containing a translate of P must have at least two connected components, with (at least) one being a translate of C_A. Since lattice animals are connected, they contain no translates of P; therefore Theorem 2.1 of [22] says that lattice animals form an exponentially small subset of entangled clusters (with respect to number of edges)—i.e., λ is strictly smaller that λ_e. □

The proof of Theorem 4, for olympic ring networks, is generally like the proof for entangled clusters in Theorem 1.

Proof of Theorem 4: The existence of the limit may be shown as in the proof for entangled clusters (Theorem 1), except we need a different form of concatenation to ensure that all components remain cycles. The procedure for concatenating cycles (self-avoiding polygons) is due to [13] and is described in Theorem 3.2.3 of [23]. The application to our case is straightforward, and we omit the details.

We now turn to the bound on the limit. Let L be an olympic ring network containing the origin and having N edges. Let A be an animal containing L that has as few edges as possible. By Corollary 16, A has at most $2N$ edges. By minimality of A, the only cycles in A are precisely the cycles of L. Clearly we can add edges to A so as to produce an animal A' with exactly $2N$ edges and no new cycles. (E.g. let B be an animal with no cycles and $2N - |E(A)|$ edges, and let $A' = A \circ B$.) Then A' contains a unique olympic ring network with N edges (namely L). The result follows since the number of possible animals A' is at most a_{2N}. □

Finally we prove the estimate on the number of lattice animals used to obtain the rigorous numerical bounds (7)–(9).

Proposition 20 *The limit* $\lambda := \lim_{n\to\infty} a_n^{1/n}$ *satisfies* $\lambda \le 5^5/4^4$.

Proof We use the method of [18, Lemma 1]. For a lattice animal A, let $b(A)$ be the number of edges of $\mathbf{Z}_G^3 \setminus A$ that have at least one endpoint in A (we call these the *boundary edges* of A). Consider bond percolation with parameter p, and let W^* be the connected cluster containing the origin. Then for every positive integer n,

$$1 \ge P_p\{|E(W^*)| = n\} = \sum_A p^n(1-p)^{b(A)} \tag{13}$$

where the sum is over all lattice animals A having n edges and containing the origin.

We claim that $b(A) \le 4n + 6$ for any lattice animal A in $\mathbf{Z}_G^3$ with n edges. To prove this, let A be an animal with n edges, and let v be the number of vertices in A. Since A is connected, we know $n \ge v - 1$ (since A contains a spanning tree, which has $v - 1$ edges). Next, consider the list of all pairs (ν, ϵ) where ν is a vertex of A and ϵ is an edge of $\mathbf{Z}_G^3$ that has ν as an endpoint. On the one hand, since each ν is an endpoint of exactly 6 edges, the list must have exactly $6v$ pairs. On the other hand, each edge of A appears twice in the list, and each boundary edge of A appears at least once in the list. Therefore $6v \ge 2n + b(A)$. Combining this with $n + 1 \ge v$ shows that $6n + 6 \ge 2n + b(A)$, and the claim follows.

Applying the claim of the preceding paragraph to (13), we obtain $1 \ge p^n(1-p)^{4n+6}a_n$. Take nth roots of both sides and let $n \to \infty$; we obtain $1 \ge p(1-p)^4\lambda$, or $\lambda \le p^{-1}(1-p)^{-4}$. This bound is optimized by setting $p = 1/5$, and this proves the proposition. □

6.2 Entanglement Percolation

The first result in this section proves Theorem 3 and is also a significant step towards the proof of Theorem 2.

Theorem 21 *For all $n \geq 1$ and all $p \in (0,1)$,*

$$P_p\{|C(\mathcal{E}_0)| \geq n\} \leq \sum_{N=n}^{\infty} e_N p^N. \tag{14}$$

For $p < 1/\lambda_e$, the probability $P_p\{|C(\mathcal{E}_0)| \geq n\}$ decays exponentially in n (indeed, (5) holds) and $P_p\{|C(\mathcal{E}_0)| = \infty\} = 0$. Hence $p_e^0 \geq 1/\lambda_e$.

Proof Fix n and let $C = C(\mathcal{E}_0)$. If $|C|$ is finite, then $C \in \mathcal{F}$. If $|C|$ is infinite, then, by definition of $\mathcal{E}_0$, there is a finite subgraph L of C such that $0 \in V(L)$, $L \in \mathcal{F}$, and $|L| \geq n$. Since C is open, so is L. Therefore,

$$\{|C(\mathcal{E}_0)| \geq n\} \subset \bigcup_{L \in \mathcal{F}, 0 \in V(L), |L| \geq n} \{L \text{ is open}\}.$$

This directly implies (14). The rest of the theorem follows from (14) and the existence of the limit λ_e from Theorem 1. □

Theorem 2 is a consequence of the following corollary and (4).

Corollary 22 $p_e^1 \geq 1/\lambda_e$.

Proof Assume $p_e^1 < 1/\lambda_e$. Choose p such that $p_e^1 < p < 1/\lambda_e$. Then $p_e^1 < p < p_e^0$ by Theorem 21. By Theorem 2 of [15], there is a finite number β such that $P_p\{|C(\mathcal{E}_0)| \geq n\} \geq \exp(-\beta n^{2/3})$. But this contradicts the exponential decay proven in Theorem 21 for $p < 1/\lambda_e$. □

Remark It is possible to prove Theorem 2 using the combinatorial methods of this paper, without appealing to [15]. The details require a few pages to explain, so we have not included them here.

7 Discussion

The main contribution of this paper is a rigorous proof that the number of entangled clusters grows at most exponentially in the number of edges. The existence of the growth constant λ_e then follows by relatively standard methods. We also obtain the explicit bound $\lambda_e \leq 4\lambda^2$, where λ is the growth constant for lattice animals (counted by edges), although we do not believe this to be a particularly good bound. We also show that the critical probability for entanglement percolation is at least $1/\lambda_e$; indeed, if $p < 1/\lambda_e$, then the distribution of the size of an entangled cluster has exponentially decaying tails.

Once we know that λ_e is finite, several methods for studying general lattice animals can be applied to entangled clusters. Here are some examples. As we observed in the proof of Theorem 1 (in Sect. 6.1), the pattern theory developed in [22] can be applied to entangled clusters. In particular, one can apply Theorem 2.2 of [22] to prove the following strengthening of our Theorem 1:

$$\lim_{N\to\infty} \frac{e_{N+1}}{e_N} = \lambda_e. \tag{15}$$

In addition, the Pattern Theorem 2.1 of [22] implies that most large entangled clusters consist of many connected components. Theorem 1.1 of [21] extends directly to entangled clusters, proving that there exists a constant K such that $e_N \le KN^{1/3}\lambda_e^N$ for all $N \ge 1$ (which improves upon the bound $e_N \le KN\lambda_e^N$ that follows from the supermultiplicative property). In a different direction, establishing the finiteness of λ_e is a first step towards a statistical mechanical analysis of ensembles of entangled clusters with specified energy functions (e.g. monomer-monomer attraction, under the formalism developed in [24]).

While writing this article, we learned that Ádám Timár had independently proven the finiteness of λ_e. However he has not yet completed the manuscript, and we have no further information about his proof. Also, after this article was accepted for publication, we learned that Grimmett and Holroyd [11] have a new improved lower bound for p_e (however, their method does not prove that λ_e is finite).

Acknowledgements N.M. is grateful to Geoffrey Grimmett and Alexander Holroyd for discussions and encouragement. We thank the referees for helpful comments. The research of N.M. is supported in part by a Discovery Grant from NSERC of Canada. N.M. would like to thank the Fields Institute for its hospitality, both when he first learned about the problem from Geoffrey Grimmett in 1998, and when this work was being completed in 2008–09.

References

1. Adler, J., Aharony, A., Blumenfeld, R., Harris, A.B.: Series study of percolation moments in general dimension. Phys. Rev. B **41**, 9183–9206 (1990)
2. Aizenman, M., Grimmett, G.: Strict monotonicity for critical points in percolation and ferromagnetic models. J. Stat. Phys. **63**, 817–835 (1991)
3. Boyd, R.H., Phillips, P.J.: The Science of Polymer Molecules. Cambridge University Press, Cambridge (1993)
4. de Gennes, P.-G.: Scaling Concepts in Polymer Physics. Cornell University Press, Ithaca (1979)
5. Diao, Y., Janse van Rensburg, E.J.: Percolation of linked circles. In: Whittington, S.G., Sumners, D.W., Lodge, T. (eds.) Topology and Geometry in Polymer Science, pp. 79–88. Springer, New York (1998)
6. Edwards, S.F., Vilgis, T.A.: The tube model theory of rubber elasticity. Rep. Prog. Phys. **51**, 243–297 (1988)
7. Gaunt, D.S., Ruskin, H.: Bond percolation in d dimensions. J. Phys. A, Math. Gen. **11**, 1369–1380 (1978)
8. Giblin, P.J.: Graphs, Surfaces and Homology, 2nd edn. Chapman and Hall, London (1977)
9. Grimmett, G.: Percolation, 2nd edn. Springer, Berlin (1999)
10. Grimmett, G.R., Holroyd, A.E.: Entanglement in percolation. Proc. Lond. Math. Soc. **81**, 485–512 (2000)
11. Grimmett, G.R., Holroyd, A.E.: Plaquettes, spheres, and entanglement. Preprint (2010)
12. Häggström, O.: Uniqueness of the infinite entangled component in three-dimensional bond percolation. Ann. Probab. **29**, 127–136 (2001)
13. Hammersley, J.M.: The number of polygons on a lattice. Proc. Camb. Philos. Soc. **57**, 516–523 (1961)
14. Holroyd, A.E.: Existence of a phase transition for entanglement percolation. Proc. Camb. Philos. Soc. **129**, 231–251 (2000)
15. Holroyd, A.E.: Inequalities in entanglement percolation. J. Stat. Phys. **109**, 317–323 (2002)
16. Janse van Rensburg, E.J.: The Statistical Mechanics of Interacting Walks, Polygons, Animals and Vesicles. Oxford University Press, Oxford (2000)
17. Kantor, Y., Hassold, G.N.: Topological entanglements in the percolation problem. Phys. Rev. Lett. **60**, 1457–1460 (1988)
18. Kesten, H.: Analyticity properties and power law estimates of functions in percolation. J. Stat. Phys. **25**, 717–756 (1981)
19. Klarner, D.A.: Cell growth problems. Can. J. Math. **19**, 851–863 (1967)
20. Klein, D.J.: Rigorous results for branched polymers with excluded volume. J. Chem. Phys. **75**, 5186–5189 (1981)
21. Madras, N.: A rigorous bound on the critical exponent for the number of lattice trees, animals, and polygons. J. Stat. Phys. **78**, 681–699 (1995)

22. Madras, N.: A pattern theorem for lattice clusters. Ann. Comb. **3**, 357–384 (1999)
23. Madras, N., Slade, G.: The Self-Avoiding Walk. Birkhäuser, Boston (1993)
24. Madras, N., Soteros, C.E., Whittington, S.G., Martin, J.L., Sykes, M.F., Flesia, S., Gaunt, D.S.: The free energy of a collapsing branched polymer. J. Phys. A, Math. Gen. **23**, 5327–5350 (1990)
25. Menshikov, M.V., Rybnikov, K.A., Volkov, S.E.: The loss of tension in an infinite membrane with holes distributed according to a Poisson law. Adv. Appl. Probab. **34**, 292–312 (2002)
26. Otto, M., Vilgis, T.A.: Topological interactions in multiply linked DNA rings. Phys. Rev. Lett. **80**, 881–884 (1998)
27. Sauvage, J.-P., Dietrich-Buchecker, C. (eds.): Molecular Catenanes, Rotaxanes and Knots: A Journey Through the World of Molecular Topology. Wiley-VCH, Weinheim (1999)
28. Schonmann, R.H.: On the behavior of some cellular automata related to bootstrap percolation. Ann. Probab. **20**, 174–193 (1992)
29. Vilgis, T.A., Otto, M.: Elasticity of entangled polymer loops: Olympic gels. Phys. Rev. E **56**, R1314–R1317 (1997)
30. Wolovsky, R.: Interlocked ring systems obtained by the metathesis reaction of cyclododecene. J. Am. Chem. Soc. **92**, 2132–2133 (1970)